arXiv:cond-mat/0603169v1 [cond-mat.other] 7 Mar 2006

Delocalization transition in an exactly solvable many-body system in Two Dimensions

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(Dated: February 6, 2008)

We study delocalization transition in a many body system in two dimension. We identify the presence of a complex vector potential that gives rise to this transition.

PACS numbers: 05.30.Jp,03.65.Ge,03.75.Kk

Non-Hermitian Hamiltonians appear in the study of quantum dots coupled to leads where it is convenient to exclude the degrees of freedom in the leads by integrating over them. This scheme results in an effective non-Hermitian Hamiltonian whose eigenstates contain an imaginary part that accounts for the decay into the leads.

In Ref. [2], Hatano and Nelson, considered a different kind of non-Hermitian Hamiltonians, one with a constant imaginary vector potential. This leads to the appearance of a first order derivative in the Hamiltonian that, as pointed out by Efetov [1] , can be assimilated to the appearance of a certain preferential direction in space. The introduction of this ingredient gives rise to new effects such as a localization transition in a disordered one dimensional system [3].

In this context, we study analytically an exactly solvable N-body system in two-dimensions, which shows a localization to delocalization transition even in the absence of random potential. The hamiltonian is non-hermitian with a complex vector potential.

We derive the ground state eigenvalue and eigenfunction for such systems and extract several interesting properties by identifying the square of the radial wave function with the joint probability distribution (JPD) of eigenvalues of the Laguerre ensembles of random matrix theory (RMT). We also study the particle density and pair correlation function (PCF) for such systems. We observe "universality" in the PCF as expected for such systems [4] from random matrix results.

Let us consider a two-dimensional system of N particles in the $x-y$ plane, subject to a vector potential $\vec{A} = [x\hat{x} + y\hat{y}]/[2(x^2+y^2)]$. The single body Hamiltonian is given by

$$H_{sp} = \left[(\hbar/i)\nabla + i\vec{A}\right]^2. \tag{1}$$

The many body Hamiltonian is given by

$$H = \sum_j \{[(\hbar/i)\nabla + i\vec{A_j}]^2, \tag{2}$$

where A_j is the vector potential.

Making the transformation $x = r\cos\theta$ and $y = r\sin\theta$ and taking $\hbar = m = 1$, we get

$$H = -\sum_i \frac{\partial^2}{\partial r_i^2} - \sum_i \frac{1}{r_i^2}(\frac{\partial^2}{\partial \theta_i^2} + \frac{1}{4}). \tag{3}$$

Here, we have used the relation

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}. \tag{4}$$

In terms of the polar coordinate, the vector potential can be written as

$$\begin{aligned} \vec{A} &= -\frac{1}{2}\nabla_\theta \hat{\theta}, \\ &= \frac{\hat{r}}{2r} \end{aligned} \tag{5}$$

where,

$$\nabla_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}, \qquad \nabla_r = \frac{\partial}{\partial r}, \tag{6}$$

and

$$\hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta, \qquad \hat{\theta} = -\hat{x}\cos\theta + \hat{y}\sin\theta, \tag{7}$$

where $\hat{x}$, $\hat{y}$, $\hat{r}$ and $\hat{\theta}$ are unit vectors along the x, y, r and θ directions respectively. This expression in turn allows us to view the complex vector potential as some velocity operator acting along the angular coordinate. This maybe the reason for which one can do away with the random part of the potential (as is seen in the work of previous authors), but retaining the effect of disorder. However, one needs to understand this aspect more clearly.

Replacing this in the Schrodinger equation

$$H\Psi_n(r_i, \theta_i) = E_n \Psi_n(r_i, \theta_i), \tag{8}$$

we obtain the ground state eigenfunction and eigenvalue.

Let us take the solution wavefunction of the form $\Psi_n(r_i, \theta_i) = R(r_1, \ldots, r_N)\Theta(\theta_1, \ldots, \theta_N)$. Using the separation of variable and equating the angular part to zero, we get

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$$-\sum_i \frac{\partial^2}{\partial r_i^2} R(r_1,\ldots,r_N) = E_n R(r_1,\ldots,r_N) \tag{9}$$

The angular part of the equation is given by

$$\frac{\partial^2}{\partial {\theta_i}^2}\Theta(\theta_1,\ldots,\theta_N) = -\frac{1}{4}\Theta \tag{10}$$

Now, let us assume that the solution wavefunction has the form

$$R_n(r_i) = \prod_{i<j}(r_i - r_j)^\lambda \exp(-\alpha_k \sum_i r_i) \tag{11}$$

$$\Theta(\theta_1,\ldots,\theta_N) = A\exp(\frac{\pm i}{2}\sum_i \theta_i) \tag{12}$$

where λ can take only two values, i.e. 0 and 1. Then $\lambda = 0$ will correspond to bosons while $\lambda = 1$ to fermions. In that case, $\alpha_k = \sum[\kappa \mp i k_n]$ and the eigenvalue for the N-body Hamiltonian will be given by

$$E_k = -N(\kappa^2 - {k_n}^2 + 2ik_n\kappa), \tag{13}$$

We must note that for a given system κ is constant. The ground state (the state that minimizes the real part of E_n) is localized for $k_n = 0$. The localized ground state is $E_0 = -N\kappa^2$. In the extended state, in order to have a well defined wave function, we need to specify the boundary conditions specific to the system. The quantity κ is a measure of the amount of localization.

Now, we will prove that (11) is indeed a solution of the radial equation (9). Let us take

$$\phi = \prod_{i<j}(r_i - r_j)^\lambda \tag{14}$$

$$\varphi = \exp(-\alpha_k \sum_i r_i) \tag{15}$$

It can be easily seen that

$$\sum_i \frac{\partial^2}{\partial r_i^2}(\phi\varphi) = \sum_i \left[\phi\frac{\partial^2\varphi}{\partial r_i^2} + 2\frac{\partial\phi}{\partial r_i}\frac{\partial\varphi}{\partial r_i} + \varphi\frac{\partial^2\phi}{\partial r_i^2}\right] \tag{16}$$

$$\tag{17}$$

Substituting Eqs.(14,15), the first term gives

$$\phi\frac{\partial^2\varphi}{\partial r_i^2} = N(\kappa^2 - {k_n}^2 + 2ik_n\kappa) \tag{18}$$

while

$$\varphi\frac{\partial^2\phi}{\partial r_i^2} = 2(\lambda^2 - \lambda)\sum_{i<j}\frac{1}{(r_i - r_j)^2}\phi\varphi. \tag{19}$$

The middle term becomes zero due to symmetry. We can also see that for Eq.[11] to be a solution of the radial part of the Hamiltonian, λ can only be 0 or 1. Thus we have proved Eq.[11]. The proof of the angular part is straightforward.

It is interesting to note from Eq.(13) that for $k_n = 0$, the wavefunction is exponentially localized and the eigenvalue does not contain any imaginary part. However, as k_n increases, the wavefunction becomes extended in nature and a complex part is added to the eigenvalue.

It is at this point that, we will write ψ_n as

$$\psi_n = C^{1/2}\prod_{i<j}|r_i - r_j|^\lambda \exp[\pm\alpha_k r_i]\exp[\pm\frac{i}{2\theta}], \tag{20}$$

Defining the scalar product (ψ_n, ψ_m) as

$$\begin{aligned}(\psi_n,\psi_m) &= C\int\int\prod|r_i - r_j|^\beta \times \\ &\quad \exp(-\kappa\sum(r_i) \pm i(k_n - k_m)\sum r_i) \\ &= \pi^N C\delta_{nm}.\end{aligned} \tag{21}$$

Here for $n \neq m$, the term on the right is highly oscillatory and hence goes to zero. For $n = m$, we get the normalization. Here, C is the well known Selsberg constant given by

$$C^{-1} = \prod_{j=0}^{N-1}\frac{\Gamma(1+\lambda+j\lambda)\Gamma(1+j\lambda)}{\Gamma(1+\lambda)}. \tag{22}$$

Then $|\psi_n|^2$ is given by

$$|\psi_n|^2 = C\prod_{i<j}|r_i - r_j|^\beta \exp(-2\kappa\sum(r_i), \tag{23}$$

with $\beta = 2\lambda$, and C being the normalization constant.

Now one may interpret $|\psi_n|^2$ to be identical with the j.p.d. of Laguerre ensembles of random matrices [4]. We define the m-particle correlation function

$$Y_m(r_1, r_2, \ldots, r_m) = \int\ldots\int dr_{m+1}\ldots dr_N d\theta_{m+1}\ldots d\theta_N |\psi_0|^2. \tag{24}$$

as the probability of finding m-particles in the intervals r_i and $r_i + \Delta r_i$ and $\theta_i + \Delta\theta_i$, irrespective of the position of the other particles. $m = 1$ and 2 correspond to the particle density and PCF respectively.

It has also been shown in Ref. [4] that $Y_1(x)$ corresponds to the density of zeros of the polynomial having weight function given by that of the Laguerre Polynomial. It has been shown by Dyson in the context of random matrices that for $\beta = 1$, 2 and 4, Y_m can be written in terms of orthogonal and skew-orthogonal polynomials. For $\beta = 2$, corresponding to the Fermionic solution, the PCF can be written as

$$Y_2(r_1, r_2) = \sum_{\mu=0}^{N-1} {h_\mu}^{-1}[q_\mu(r_1)q_\mu(r_2)]\exp[-2\kappa r_1] \tag{25}$$

where $q_\mu(r)$ are orthogonal polynomials corresponding to the normalization condition

$$\int_{-\infty}^{\infty} q_\mu(y) q_\nu(y) \exp[-2\kappa r] dy = h_\mu \delta_{\mu\nu}. \tag{26}$$

It should be noted that our choice of circular coordinate has considerably simplified our calculation as $|\psi|^2$ does not contain θ explicitly. Using the Christoffel-Darboux formula in Eq.[25], we can perform the sum. Finally using the asymptotic form of the Laguerre polynomials, and taking the limit, we get for the particle density. Here, we will give an alternate derivation (used previously in [4]) for the derivation of particle density.

We define the resolvent

$$G(z) = \int \frac{Y_1(r)}{z-r} dr, \tag{27}$$

which satisfies

$$G(x+i0) = \int \frac{Y_1(r)}{x-r} dr - i\pi Y_1(r). \tag{28}$$

Also, from the definition of $|\psi_n|^2$, we may write [4]

$$\frac{\partial Y_1(r)}{\partial r} \;=\; \beta \int \frac{Y_2(r,r_2)}{(r-r_2)} dr_2 - 2\kappa Y_1(r). \tag{29}$$

For large N, replacing $Y_2(r,r_2) \simeq Y_1(r) Y_1(r_2)$ and dropping $\partial Y_1(r)/\partial r$, we get

$$-2\kappa \int dr \frac{r Y_1(r)}{(z-r)} + \frac{\beta}{2} \int\int dr dr_2 \frac{z Y_1(r) Y_1(r_2)}{(z-r)(z-r_2)} = 0. \tag{30}$$

This gives

$$\beta z G^2(z) + 4\kappa N - 4\kappa Z G(z) = 0, \tag{31}$$

solving which we get for the particle density

$$\pi\beta Y_1(r) \;=\; 2\kappa \sqrt{\frac{N\beta/\kappa - r}{r}}. \tag{32}$$

It is interesting to note that as pointed out earlier, κ is a measure of the degree of localization. For κ large compared to N, particles get more strongly localized around the origin. However, as κ decreases, particles start getting delocalized, until they form extended states. Also, it should be noted that in the thermodynamic limit, the particle density, after proper scaling, can be made independent of β and hence applicable for both boson and fermion. However, in this case, the nature of localization is exponential for bosons ($\lambda = 0$), while it is given by Eq.[32] for fermions ($\lambda = 1$).

To calculate the PCF, essential for the study of thermodynamic properties, for $\beta = 2$, it has been proved in Ref.[4] that for interparticle spacing $(r_1 - r_2) \equiv \Delta r \to 0$, $N \to \infty$ and defining $r' = \Delta r . Y_1(r)$, the scaled PCF is universal. For $\beta = 2$, it is given by

$$Y_2(r_1, r_2) = 1 - \frac{\sin^2(\pi r')}{(\pi r')^2}. \tag{33}$$

This shows that the probability of finding two particles in the same state ($\Delta r = 0$) is zero. This is in agreement with the Pauli exclusion principal, applicable for a system of fermions.

Thus we have studied the localization to delocalization transition in a system of N-particles in the presence of a complex vector potential. For localized state, particles tend to gather around the origin, as shown in Eq.[32]. At the same time, the PCF shows that the particles are not allowed to condense to one position. The corresponding eigenvalue is real. However, for delocalized state, we observe that the eigenvalue has complex part.

S.G. is grateful to Dr. Louis Foe Torres for many useful discussions.

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